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Force Dynamics of Tempo Change in Music

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This paper reports a study of the quantitative shape of ideally executed tempo decrease and increase (ritardando and accelerando) in music. Five expertly performed samples of real music (two ritards and two accelerandos from European classical music, and one sample of Venezuelan [Yanomami] Indian barter chant) are analyzed, and each is found to be a polynomial of degree at least two, and in some cases probably three—not simply linear as had been previously reported, or exponential, as is sometimes suspected. We formally develop a simple force model of musical "motion," in which the progression of music over time is conceived of as being controlled by a mental analog of a mechanical force. Each of the five observed tempo profiles is shown to be well accounted for as the result of one of just two types of force event: (1) a linear force, which produces a quadratic ritardando (or accelerando), and (2) a parabolic force, which produces a spline-shaped cubic tempo profile, a ritardando that joins smoothly at both ends to adjoining regions of constant tempo.

Introduction: Tempo Change in Music

Change of tempo in music is diagnostic of the interplay between mental processes (cognitive, affective, and so forth) and formal musical structures, since control over the exact pacing and mathematical pattern of a tempo change is not normally notated explicitly in the score, but rather is ceded completely to the intuitive judgment—the "ear"—of the performing musician. The slowing of the beat in a ritardando or the speeding up in an accelerando, for example, signal purely mental phenomena. Moreover, the control of musical timing must rest at a somewhat more abstract level than that of an ordinary motor schema, such as the one that controls the swing of a tennis racket. A long ritard, for instance, might require a very large number of independent constituent motions to execute; a subset of these contribute control to the timing of the ritard (i.e., those that produce a note), but many others do not (e.g., bow recovery in stringed instruments). Yet the skilled musician can keep track of the overall pacing of the

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ongoing music largely independently of the pacing of most of these individual motions, determining whether it is conforming correctly to his or her abstract mental model of a "natural-sounding" or "musical-sounding" ritard.

We investigate the mathematical shape of the tempo profile in ideally executed tempo change, in an effort to attach a quantitative meaning to the idea "natural sounding." We are concerned in particular with the similarity between actual expert performance of ritards and hypothetical abstract ideal forms that we will derive from a formalization of musical flow. Our presumption is that the consummate aesthetic proportion achieved in the expert samples reflects a match with some kind of ideal mental model of a ritard that the performers share, at an unconscious level, with listeners.

In order that we may consider the shape of a hypothetical tempo curve, we must assume that the discrete beat durations produced by the musician were generated by sampling (at discrete intervals along the time axis, namely the indicated attack points of the beats) of some mostly continuous function of time representing the musical tempo. Normal constant-tempo music then naturally corresponds to a constant function (whose value is the tempo) while new tempos correspond to step discontinuities. But what shape does this curve take when the tempo changes gradually, as in a ritardando or accelerando?

Sundberg and Verrillo (1980), analyzing very brief passages of ritards, concluded that the shape of their tempo profile is linear. This model seems surprising, however, because then a ritard could never join smoothly with adjacent tempos, as might be aesthetically desirable; the onset of a ritard would always produce a discontinuity in the first derivative of the tempo profile, which might be heard as a "jerk." A belief sometimes expressed by musicians is that the shape of the ritard curves is exponential, perhaps as this model brings to mind a natural growth process; however, here again a smooth join at both ends is impossible. Whether a smooth join is desirable in a particular musical context depends, of course, on the aesthetic judgment of the performer—although the composer's marking poco a poco ("little by little") that frequently accompanies ritards and accelerandos seems to call for a smooth join nearly explicitly (at least at the front end). Our concern, rather, is with characterizing the formal planning apparatus that must be available to the performer in order that a smooth join may be achieved if the music seems to call for it.

To obtain a smooth initiation of a tempo change (i.e., the derivative is continuous at the join with a constant previous tempo), a polynomial of at least degree two is required; to obtain smooth joins with both adjoining tempos (so that the tempo profile's derivative is continuous at both the beginning and the end of the tempo change), a polynomial of degree three is required.[1] Cubic polynomials play a critical role in the theory of motor control; they minimize the "jerk" or "jumpiness" of limb movements (see for example Atkeson, An, & Hollerbach, 1988; Flash & Hogan, 1985; Richards, 1988). In much the same way, musical ritards executed to the profile of a cubic spline would satisfy a natural constraint of smooth, graceful gesture with the minimal formal apparatus. It may be that ritards longer than Sundberg and Verrillo examined (i.e., yielding more data) are required in order to reveal such higher-order trends. Hence in this paper we examine five much longer musical samples, long enough to detect a cubic component in the tempo profile if one is present.

The Force Model

In this section, we translate into mathematical terms an intuition, common among performing musicians, that changing the tempo of the musical beat involves something like the application of a force to the beat, as evidenced by such figures of speech as "pushing forward," for an accelerando, and "holding back," for a ritard. This notion is consistent with the commonplace conception of musical progression through time as a sort of movement, one that proceeds at a constant pace unless it is impelled to do otherwise by the performer.

The progression of music through time can be conceived of as the movement along the time axis of a (purely abstract) "particle of music," whose position at time t is given by monotonically nondecreasing $P = P(t)$. The derivative of the particle's position with respect to time, $H(t) = \dot{P}(t)$, is a function representing the tempo of the music at time t, whose inverse is the duration of a beat starting at time t. $P(t)$ may also be conceived of purely "formally" as $\int H(t)dt$, under the constraint $H(t) \geq 0$ for all t. The reader may find this latter construction somewhat more intuitive than the position of the particle at time t, to which it is difficult to attach any physical interpretation (but see footnote 2). The particle has some "mass" m, which may be thought of naturally as the resistance of the music to change in tempo (depending, we might imagine, on musical factors such as the emotional "heaviness" of the passage, or on performance parameters such as the ease of maintaining ensemble); or else, again, as a purely formal parameter. Throughout the following analysis, it should be kept

1. The degree comes from four boundary conditions: the initial and final tempos, and the initial and final rates of change, both zero. These four conditions then suffice to solve for the four constants that define the cubic polynomial. A polynomial used to meet boundary conditions in this fashion is commonly called a "spline."

in mind that the "forces" we are discussing are not physical but rather mental components of some plan of action residing in the mind of the musical performer.[2]

Following a conjugate of Newton's second law $F = ma$, the particle responds to a time-varying force $F(t)$, according to

$$F(t) = m\ddot{P}(t). \tag{1}$$

This domain actually makes a purer instantiation of Newtonian theory than do physical objects, in a sense, because of the total absence of "friction." In the next section, we measure beat durations of actual musical passages, so the governing equation is better expressed as

$$H(t) = \dot{P}(t) = \dot{P}(t_0) + \frac{1}{m}\int_{t_0}^{t} F(t)dt, \tag{2}$$

where $\dot{P}(t_0)$ is the initial tempo at time t_0 before the initiation of the force.[3] Because we are interested in only the local shape of the tempo change event itself, we restrict our attention to the unit interval [0,1] and assume without loss of generality that the initial tempo $\dot{P}(t_0) = 0$, and $m = 1$ and hence drops out. In this case Eq. (2) simplifies to

$$H(t) = \int F(t)dt, \tag{3}$$

that is, a simple integral relationship between the force profile and the tempo profile, so that the tempo event is simply the formal integration of the force event. This relationship enforces a difference of one in polynomial degree between any force event and the corresponding tempo event. A null (everywhere zero) force event yields a constant tempo event, constant force yields linear tempo, linear force yields quadratic tempo, and so forth. Enumeration of these simple force event types leads to a very short list of basic tempo profiles, in a manner quite similar to the enu-

2. Interestingly, at least one physical system (nearly) instantiates this theory literally, rather than just "mentally," as the musical performer putatively does: the mechanism of a music box. This device typically consists of a cylinder with raised bumps, which rotates on its axis, causing the bumps to pluck metal spokes, which produce tones. Assume the music is encoded on the cylinder perfectly "straight" (i.e., without any tempo inflection), and the cylinder turns without any friction (which of course it cannot really). Then the force equation governing the rotation of the cylinder, $N(t) = I\ddot{\theta}(t)$ (relating torque, moment of inertia, and angular acceleration), is subject to the same treatment of angular velocity change as is linear velocity in the present theory, and the resulting music would consequently be subject to the identical categorization of tempo change events.

3. Note that the sign (+/−) here is unimportant; for each type of ritard there is a corresponding accelerando. For simplicity we will henceforth assume an increasing tempo profile, corresponding to a ritard if the ordinate is interpreted as beat duration. Note also the minor linguistic confusion, since the term "increase" really means an increase in beat duration, and hence a decrease in "tempo" as normally expressed.

meration of visual cues to categories of object motion constructed by Rubin and Richards (1985).

The most rudimentary tempo "event," normal constant-tempo music, thus corresponds straightforwardly to the null (everywhere zero) force, which integrates (via Eq. 3) to a constant profile. Similarly, instantaneous tempo changes can be produced by the instantaneous application of infinite force, the so-called "impulse" function, $\delta(t) = \infty$ at $t = 0$, $\delta(t) = 0$ elsewhere. This function definitionally has the property

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

(see Bracewell, 1965). Consequently, if the desired amount of tempo change is G, the force function $F(t) = G\delta(t)$ yields an instantaneous step change in the tempo function $H(t)$, analogous to a hockey stick striking the puck in a "slap shot." The instantaneous tempo event is thus

$$H(t) = \begin{Bmatrix} 0: & t < 0 \\ G: & t \geq 0 \end{Bmatrix}. \tag{4}$$

An impulse force event produces a start or stop of the music (when the prior tempo or the new tempo is zero) or a discrete tempo change (otherwise), as would be stipulated by a new tempo marking in the score.

The need for tempo continuity, in particular with an adjacent region of constant tempo as mentioned earlier, is met by two qualitative types of nonlinear tempo change: one for continuity at only one end and one for continuity at both initiation and termination.[4]

LINEAR FORCE, QUADRATIC TEMPO

Because the force event is the derivative of the tempo event, smoothness in the tempo corresponds to continuity in the force. To obtain a tempo change that begins smoothly, therefore, at least a linear force event, entailing a quadratic tempo profile, is required. If the tempo change is to be applied to normal constant-tempo music, this linear force must start at zero. Hence the simplest case of tempo change is a simple linear increase from zero to some nonzero endpoint representing the final rate of tempo change, hence $F(t) = \alpha t$, where α is the rate at which the force is increased (i.e., the slope of the line). The corresponding tempo event is thus a

4. Needless to say, these two types do not exhaust the field of possible qualitative types of tempo change. A variety of other types, including pauses (e.g., *luftpause,* breath), and rubato, can be accounted for neatly by the force model as special cases. We omit details because the musical samples considered in this paper, long passages of ritardando and accelerando, do not include examples of these types for empirical corroboration. These two types, at any rate, are not normally sustained over sufficiently long runs of notes to have their qualitative structure investigated sensitively (statistically) in vivo.

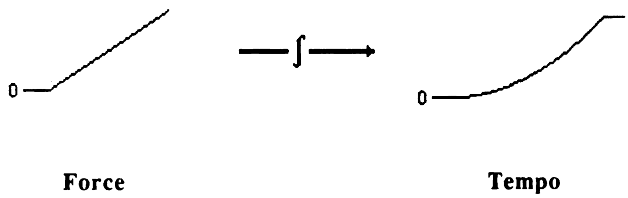

Fig. 1. A linear force (left) produces (integrates to yield) a quadratic tempo profile (right).

parabolic increase in tempo, joining the adjacent region of constant tempo smoothly, but then increasing rapidly at the other end. In formal terms,

$$H(t) = \int F(t)dt = \int \alpha t dt = \frac{\alpha}{2}t^2. \tag{5}$$

This force and tempo event is drawn schematically in Figure 1.

QUADRATIC FORCE, CUBIC TEMPO

If the linear force event discussed earlier is translated down the *y*-axis until half of it is above and half of it below zero (i.e., down a distance $\alpha/2$), we obtain a linear function that rises from below the axis on the left to above the axis on the right. When this function is inverted and then integrated, it yields a force function that, when integrated in turn, yields a tempo function that is flat at both ends (see Figure 2). As discussed earlier, such a tempo profile could join smoothly with both prior and succeeding regions of constant tempo, making a ritard (or accelerando) that begins and ends without any "jerk" in the musical gesture. We mention the linear precursor to the force function in this case both to illustrate its relationship to the linear force function of the previous section, and also because in the next section we will observe this precursor in the derivatives of the musical samples. In formal terms,

$$F(t) = \int [\alpha t - \frac{\alpha}{2}]dt = [\frac{\alpha}{2}t^2 - \frac{\alpha}{2}t], \tag{6}$$

and

$$H(t) = \int F(t)dt = \int [\frac{\alpha}{2}t^2 - \frac{\alpha}{2}t]dt = [\frac{\alpha}{6}t^3 - \frac{\alpha}{4}t^2]. \tag{7}$$

Note that in both of the above integrations, our exclusive interest in the shape of the tempo profile and not its height (i.e., not the absolute tempo) allows us to ignore the constant of integration.

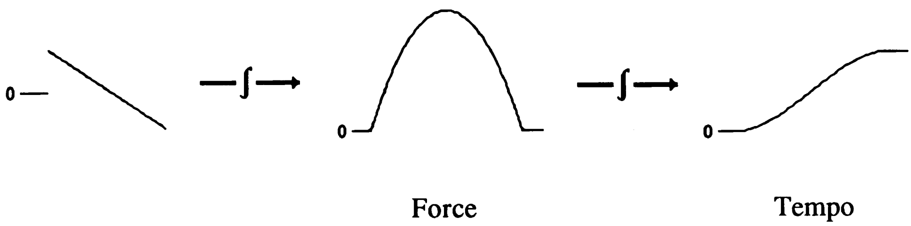

Fig. 2. A linear shape with equal parts above and below zero (left) integrates to yield a symmetric parabolic force (middle), which in turn integrates to yield a smoothly increasing tempo profile (right). The tempo event, like a cubic spline, is flat at both ends, and hence joins smoothly with both adjoining regions of constant tempo. (The precursor shape on the left, while neither a force nor a tempo, is to be compared with the derivatives of the observed data discussed in the text.)

Analysis of the Tempo Profiles in Five Musical Samples

SAMPLES

In a forthcoming book (Epstein, in press) David Epstein considers four passages: ritardandos from Dvořák and Stravinsky and accelerandos from De Falla and Tchaikovsky. The passages were selected for their unusual length, so that the time curve might be examined over as large a sample as possible, in order to permit the statistical detection of higher-order trends. For each passage, Epstein auditioned a number of performances, culled from commercially available recordings, including some of the world's most widely respected conductors. Each one of the auditioned performances already represents a highly special sample, since it presumably represents the conductor's first choice from a number of recorded "takes." From these recordings, for each piece, Epstein selected the one he judged to capture the optimally natural, musical execution of the passage. These data may thus epitomize, as closely as possible, "ideal" natural tempo change.[5]

Additionally, in an earlier paper Epstein (1985) examined a roughly 35-min sample of the barter chant ("Himou") of the Yanomami Indians of Venezuela, a rhythmic ritual conversation whose tempo fluctuated, mostly accelerating, until a bargain agreement was reached. This sample, in addition to yielding a much larger data set on which to test the theory, provides a modicum of cross-cultural corroboration.

5. These data thus represent an attempt to examine idealized, rather than average, performance. To characterize *typical* tempo change (as executed by, say, randomly selected trained musicians) would be quite interesting, but it would be a different study.

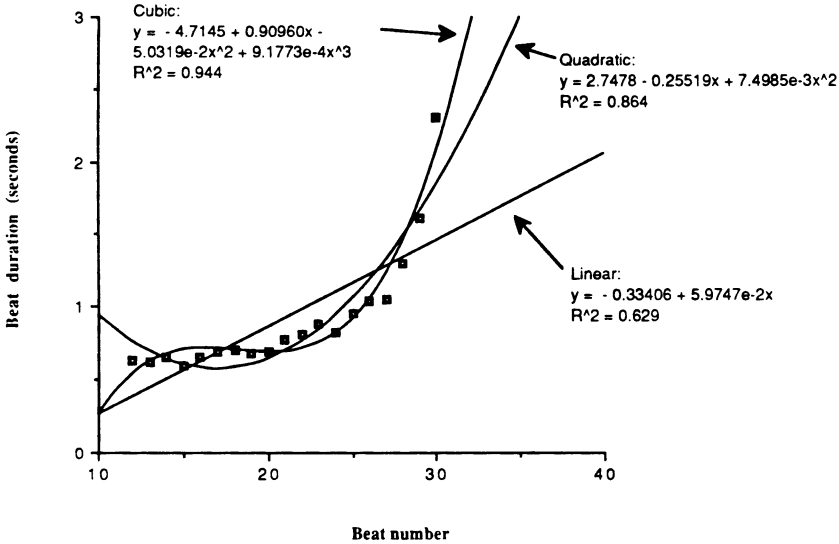

Fig. 3. Sample 1 (Dvořák, Slavonic Dance op. 46 no. 8, mm. 243–272, conducted by George Szell. Columbia/Odyssey Stereo Cassette YT 34626), showing linear, quadratic, and cubic best fit regressions plotted, with equations and degrees of fit (R^2) indicated.

The beat durations in the samples were measured and recorded, using the (rapidly played) measure as the beat in all but the very slow Stravinsky passage, in which each quarter-note beat was a measured unit. The beats were measured by first transferring the recordings to 7.5 in./sec open-reel tape. The attack points were then marked on the tape using a conventional editing technique, in which the tape is run back and forth over the playback head by hand at very slow speed until the attack point can be identified exactly. The distances between the marked points were measured to the nearest millimeter, resulting in a final resolution of about 5 msec. The time courses of the five samples are displayed graphically in Figures 3–7.

To test the fit of the polynomial models to the data, regressions to linear, quadratic, and cubic polynomials were performed. The regression curves are plotted (and the equations are indicated) in the graphs in Figures 3–7. Ideally, one would like to determine the true polynomial degree of each curve by testing the significance of the incremental contribution to the variance accounted for by the addition of successive higher-order terms. Unfortunately, however, the very large proportion of variance accounted for by these regressions, especially of the quadratic and cubic models, usually well over 90%, makes the use of significance testing problematic. The error term in such an analysis is the variance still unaccounted for with the added polynomial term, which here comes so close to zero that F-values are inflated, with the result that successive higher-order regressions almost always come out spuriously significant.

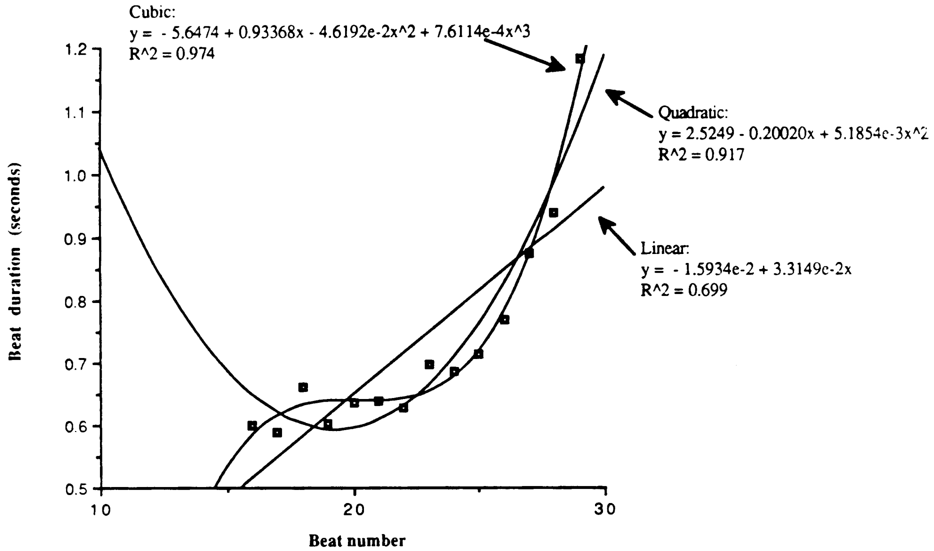

Fig. 4. Sample 2 (Stravinsky, excerpt from The Firebird (complete ballet), rehearsal number 19, mm. 12–17, conducted by the composer. Columbia Records Stereo MS 6328) showing linear, quadratic, and cubic best.fit regressions plotted, with equations and degrees of fit (R^2) indicated.

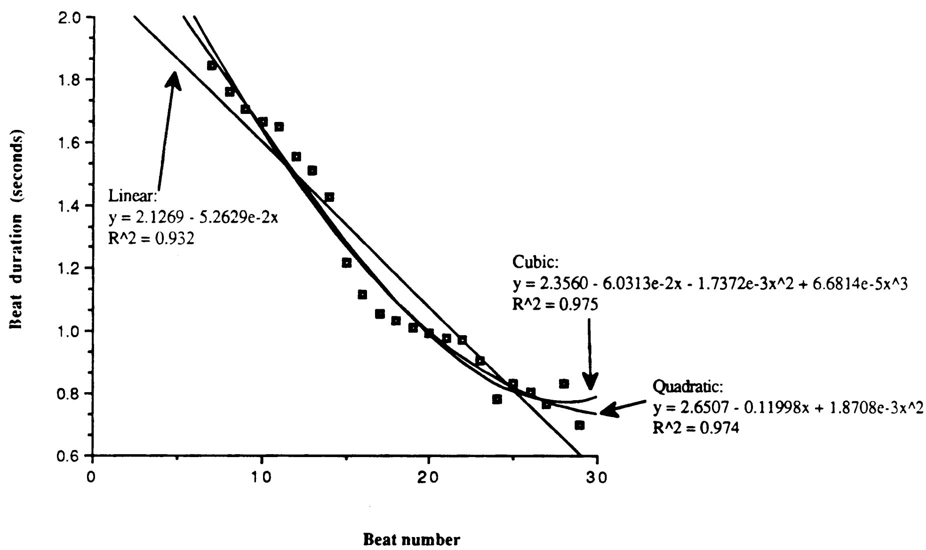

Fig. 5. Sample 3 (De Falla, excerpt from The Three-Cornered Hat (suite II), The Miller's Dance, rehearsal number 9, conducted by Herrera de la Fuente. Vox Cum Laude Record 9047), showing linear, quadratic, and cubic best fit regressions plotted, with equations and degrees of fit (R^2) indicated.

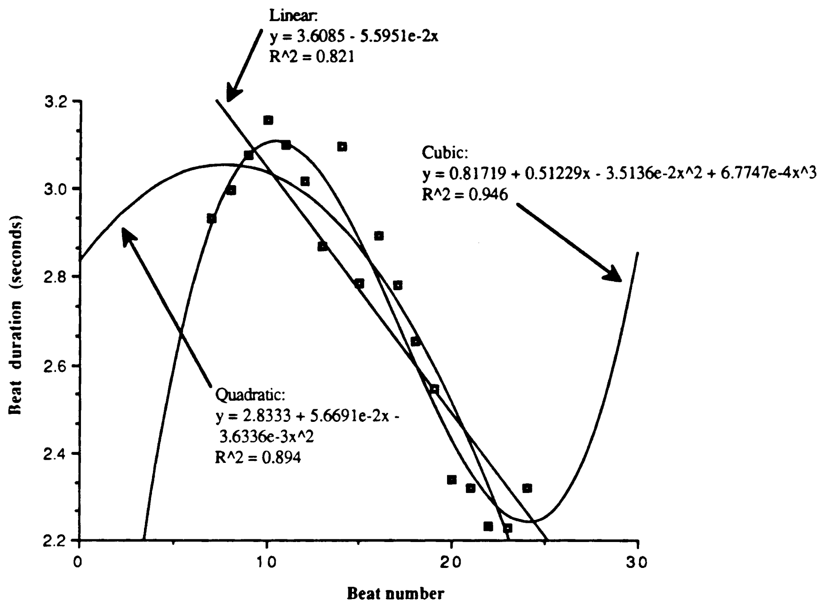

Fig. 6. Sample 4 (Tchaikovsky, excerpt from Symphony no. 4, first movement, mm. 144–161, conducted by Herbert von Karajan. Deutsche Gramophon Digital 415 348–4), showing linear, quadratic, and cubic best fit regressions plotted, with equations and degrees of fit (R^2) indicated.

The R^2 values are presented in Table 1. In every case but Sample 3 (discussed later), the proportion of variance accounted for by the linear model is substantially lower than that accounted for by the quadratic or cubic model. To the eye, more importantly, the observed profiles do not appear to be linear: in each case (again with the possible exception of Sample 3) there is a strong curved component, which in most cases flattens out at one or both ends so that its tangent joins smoothly with that of the adjacent region of constant tempo. The locations at which the profiles flattened out were generally in accord with the composers' marked tempo directions; the shapes will be discussed one by one next.

The exponential model also performed poorly. In four of the five samples, the exponential model was slightly better than the linear one, but worse than either the quadratic or cubic model; in the other case the exponential model was even worse than the linear model. In all cases, the best fit exponential appeared to be primarily linear anyway, indicating that the curvature present in the data was not exponential in nature (that is, the exponential curve fit the data only to the extent that it matched the linear best fit).

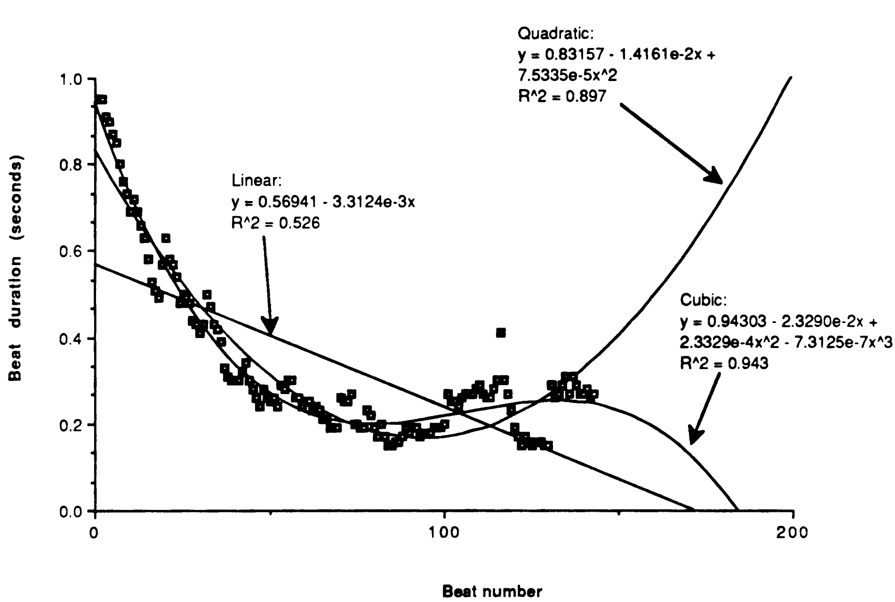

Fig. 7. Sample 5 (Himou bargain chant of the Yanomami Indians. See Epstein, 1985), showing linear, quadratic, and cubic best fit regressions plotted, with equations and degrees of fit (R^2) indicated.

In order to determine more precisely the true degree of each of these observed profiles, we numerically estimated derivatives (with respect to time) of the observed data and compared them with the derivative shapes predicted by the force model. Because the force model makes a prediction not only about the shape of the final tempo profile but also of its derivative—the force profile—and, similarly, any lower order derivatives, it is possible to "unpeel" the observed data, polynomial degree by polynomial degree, by taking successive derivatives, to see what kind of a curve it really is. Differentiation was performed by first smoothing the data with a discrete smoothing operator (of width 5 for the first four examples, and width 50 for the much longer fifth example). Then, each successive de-

TABLE 1
R^2 for Best Fit Linear, Quadratic, and Cubic Polynomials

Sample	Linear	Quadratic	Cubic	Exponential
1	.629	.864	.944	.787
2	.699	.917	.974	.765
3	.932	.974	.975	.960
4	.821	.894	.946	.818
5	.526	.897	.943	.532

rivative was taken by a discrete forward difference operator (of width 2 for the first four samples, and 10 for the fifth sample; all mask widths were chosen by eye to produce cleanest output). Derivatives down to (but not including) the first apparently zero output are shown in Figures 8–12, shown lined up with the raw beat durations. In each case, either the third or fourth derivative was apparently zero, at least over a large central portion of the sample, indicating polynomials of degree 2 or 3 in each case. Theoretically, the derivative before the first zero would be a constant, and hence nonzero, but in practice in each case the value of the constants was below the noise floor and hence apparently zero, so actually the last displayed derivative in each case is the presumably linear one.

We now proceed to examine the five samples one by one, considering both the general similarity of the linear, quadratic, and cubic best fit curves (Figures 3–7) to the shapes predicted by the force model, and also the similarity of its derivatives (Figures 8–12, in which the raw data are reproduced for side-by-side comparison) to the appropriate predicted force profiles.

SAMPLE 1 (DVOŘÁK)

Examining Figure 3, this ritard is clearly parabolic, possibly with a cubic component. Its first derivative (Figure 8b) appears to be parabolic and its second (Figure 8c) linear (except for two presumably spurious points at the end), suggesting that the cubic component in the tempo profile is actually important. In either case, the tangent of the curve flattens out near the beginning of the ritard, indicating a smooth join with the previously prevailing tempo, and thereafter rises in a nonlinear manner. There is no smooth join at the end of the passage, presumably because at this point the composer indicates a sudden discontinuous jump back to the original tempo.

SAMPLE 2 (STRAVINSKY)

This ritard (Figure 4) is nearly identical in shape to the previous sample. Again, the shape of the plotted regressions reveals a parabola, and the derivatives (Figure 9) suggest that the cubic component may be important. Again, the tangent of the curve flattens out near the beginning, indicating a smooth join with the previous section of music. At the other end, Stravinsky marks a new tempo (which is initially so slow that there is no longer any apparent beat or forward momentum), so that as in Sample 1 there is no occasion for a smooth join to a new tempo.

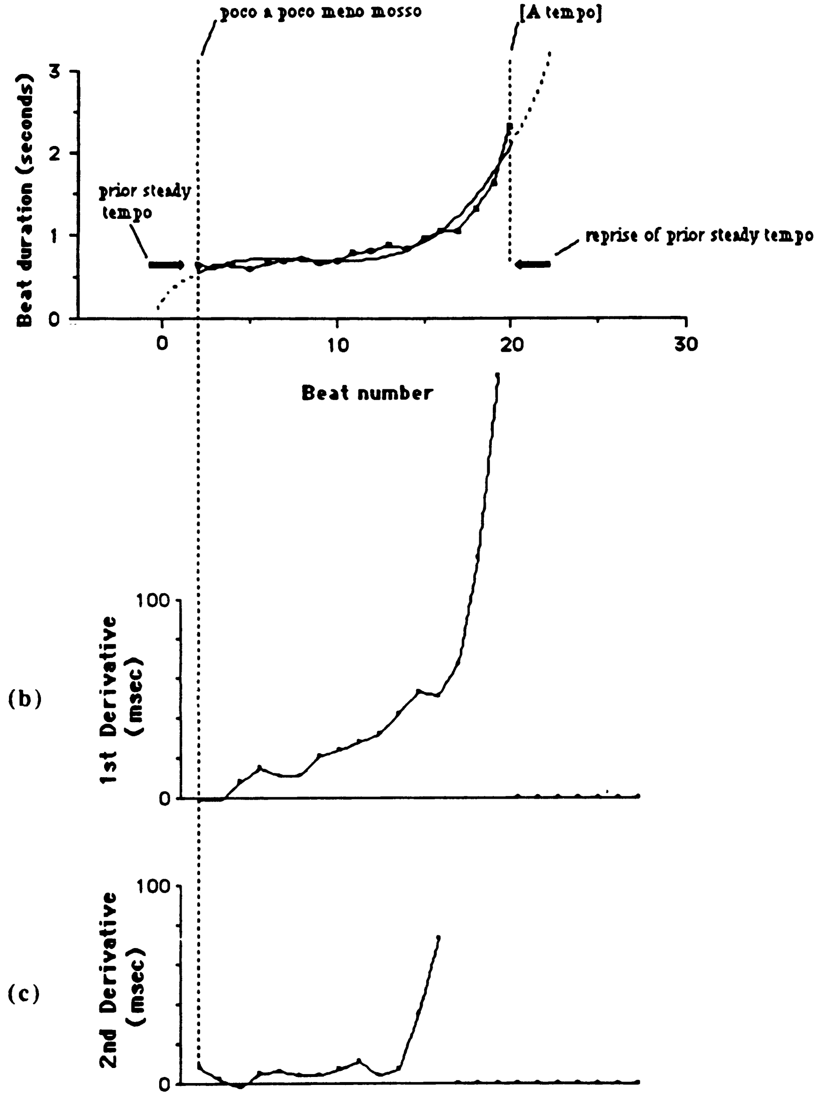

Fig. 8. Sample 1 (Dvořák), showing (a) original beat durations with best cubic fit plotted for comparison, and adjoining tempos and composer's markings indicated; (b) first derivative; (c) second derivative.

SAMPLE 3 (DE FALLA)

This passage, an accelerando, is somewhat problematic. The plot (Figure 5) suggests a nonlinear curve of possibly higher order, as the points wander in an extremely orderly manner above and below the linear regression line. This complexity is in part due to the two small step

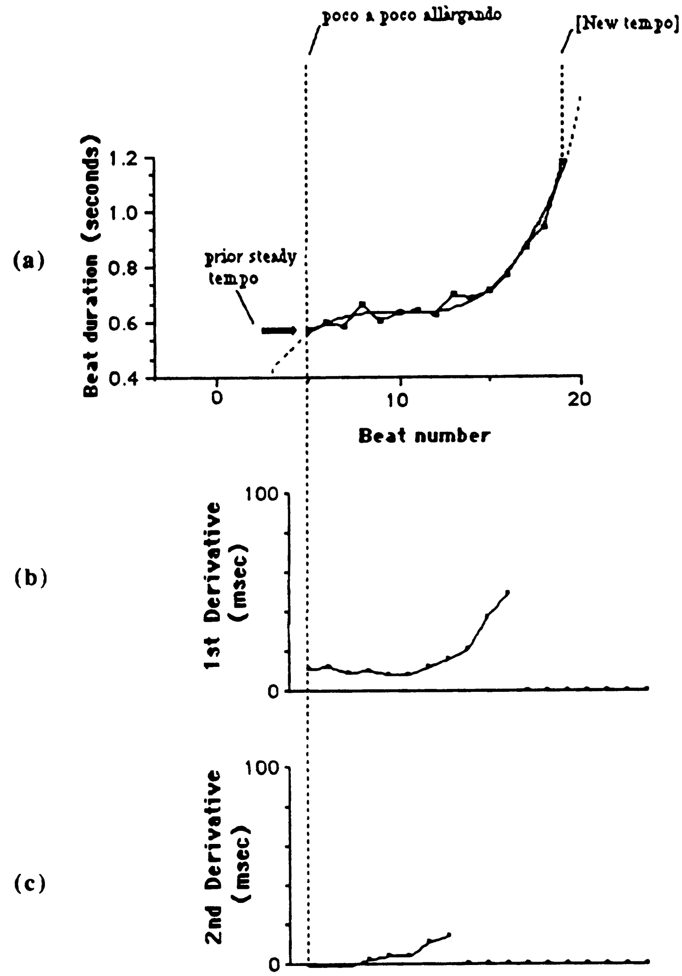

Fig. 9. Sample 2 (Stravinsky), showing (a) original beat durations with best cubic fit plotted for comparison, and adjoining tempos and composer's markings indicated; (b) first derivative; (c) second derivative.

discontinuities (marked "more lively") that the composer has indicated at even intervals in the middle of the passage. The first derivative (Figure 10b) on the other hand, manifests the characteristic central hump of the derivative of the cubic spline (Figure 2), except for the last few points. Correspondingly, the second derivative rises linearly from below the axis to above, at any rate over part of its range, as expected. The musical context of this passage is unique among our samples in that at the end of the accelerando the piece comes to a full stop. There is thus no later

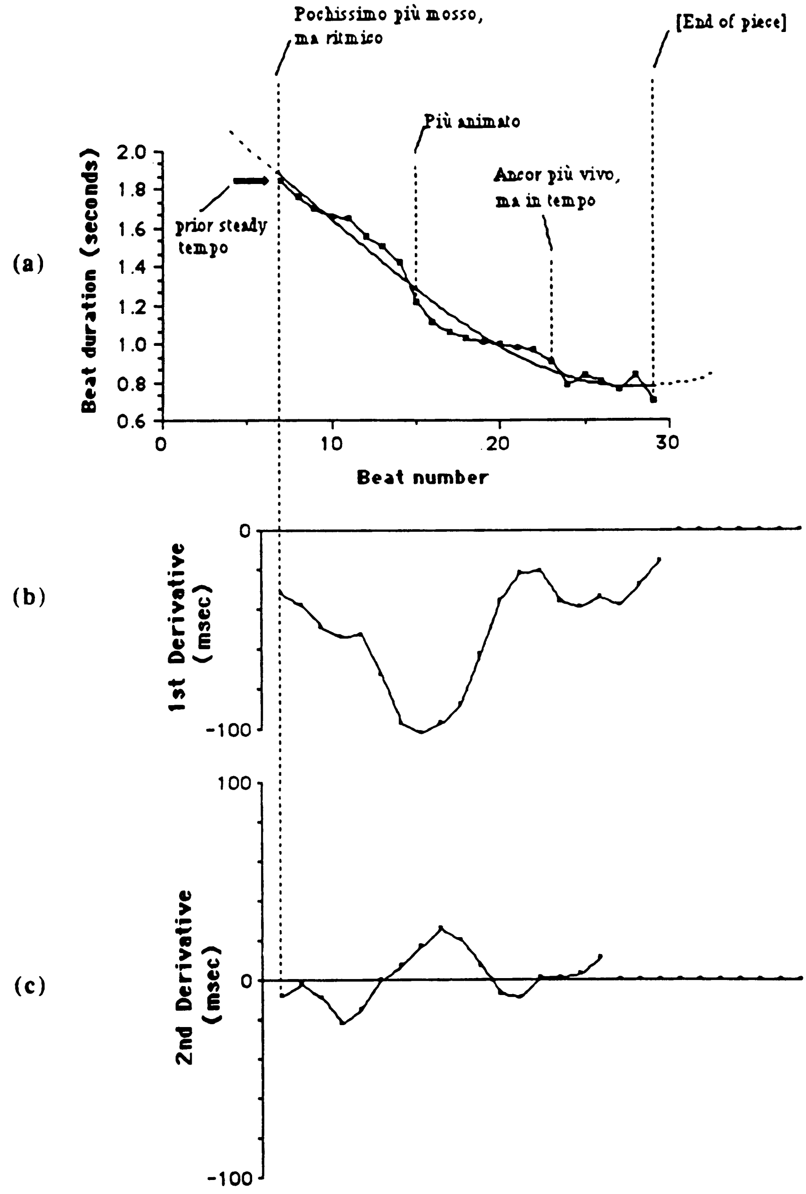

Fig. 10. Sample 3 (De Falla), showing (a) original beat durations with best cubic fit plotted for comparison, and adjoining tempos and composer's markings indicated; (b) first derivative; (c) second derivative.

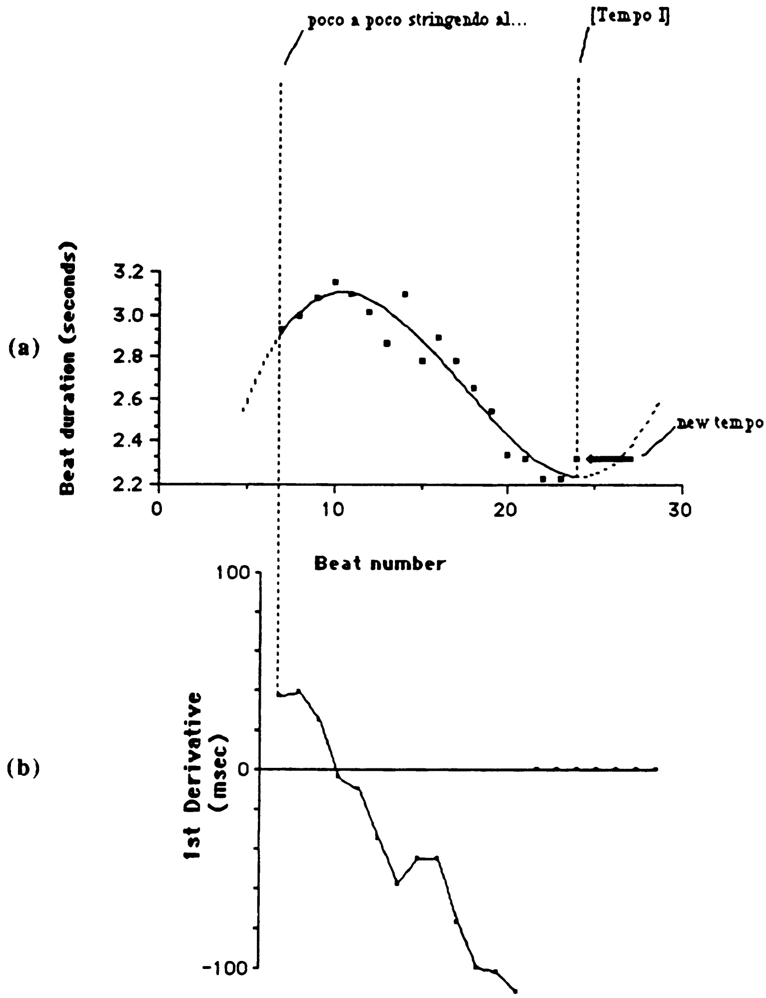

Fig. 11. Sample 4 (Tchaikovsky), showing (a) original beat durations with best cubic fit plotted for comparison, and adjoining tempos and composer's markings indicted; (b) first derivative.

section of constant tempo to which this passage needs to join; rather, the conductor's goal tempo here seems to be an integral multiple (3:1) of the initially prevailing tempo.[6]

6. Integral tempo ratios are apparently a ubiquitous and important phenomenon, and they are considered more fully in both Epstein (1979) and Epstein (in press).

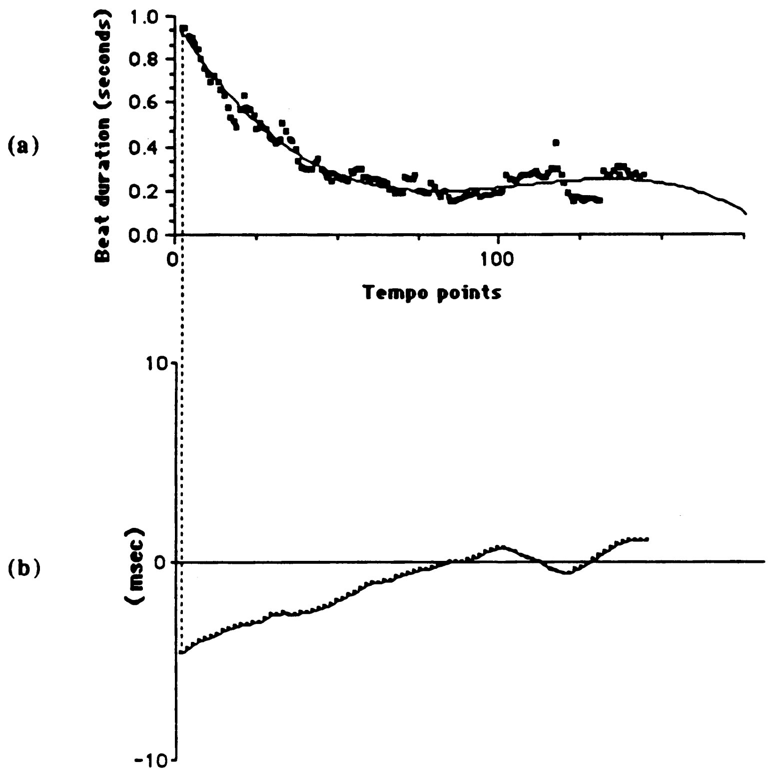

Fig. 12. Sample 5 (Himou), showing (a) original beat durations with best cubic fit plotted for comparison, and adjoining tempos and composer's markings indicated; (b) first derivative.

SAMPLE 4 (TCHAIKOVSKY)

This accelerando (Figure 6) is the only sample that unambiguously seems to require the smooth spline, in that there is ongoing music in both adjacent regions, with the same basic pulse unit but at different tempos. Exactly as expected, then, its profile has a conspicuous curvature component that flattens out near both ends. The first derivative (Figure 11b) is clearly linear, confirming the nonlinearity in the tempo profile, but casting some doubt on the cubic component. In either case, though, the profile is not linear.

SAMPLE 5 (HIMOU)

This sample (Figure 7) is clearly parabolic, starting with a steep negative tangent and flattening out at the end. The almost perfectly linear first derivative (Figure 12a) corroborates this model. The similarity of the derivative with the previous sample is striking. It is intriguing that in two such sharply contrasting musical and cultural traditions as European music and Yanomami chant protocols, a nearly identical form was produced.

All five samples, in summary, exhibit a primarily nonlinear shape. The problematic Falla (Sample 3), while not fitting any simple low-order type, clearly curves above and below the best fit straight line in an orderly although more complex manner, visibly following the composer's more complex markings. The other four examples and their derivatives all have shapes that are consistent with simple low-order force events, where the force is either linear or a simple quadratic form, and the resulting tempo is either parabolic or spline-shaped. The force model thus accounts neatly for the observed tempo profiles.

Conclusion

Musical tempo might, in principle, be changed by any arbitrary continuous function with different starting and ending values. The performer is not obligated to follow constraints such as those suggested in this paper: no physical laws, for example, require it. Yet while change of tempo by skilled musicians may *seem* to be under conscious, voluntary control, the matches found here between observed profiles and theoretical ideal shapes suggest that performers actually conform in detail to mathematical constraints of which they presumably have no conscious knowledge. If this conformance actually were responsible for the aesthetically satisfying form achieved by skilled musicians, as we speculate, it suggests a paradigm for "beauty," or at least naturalness of form, that is refreshingly concrete. Furthermore, as we have discussed, because these ideal shapes were derived from the force model, this conformance seems to reflect an underlying, unconscious conception of music as a quasi-physical thing that "moves forward" as it unfolds through time, now speeding up and now slowing down, in accord with the moment-to-moment flux in its rhythmic, harmonic, and affective character—a conception reflected in musicians' common use of terms such as "movement," "motion," and "flow" to characterize the progression of music. We propose, in effect, that the relationship between tempo change (designated by the score and manipulated by the performer) and real physical movement (controlled by physical forces) is somewhat more than just a metaphor: the formal machinery is largely the same. The "forces" we invoke are neither actual physical forces, of course, nor are they literally due to physical constraints such as those that control the movement of a conductor's baton or a drummer's stick. That these mental analogs of force actually obey constraints as if they *were* physical is thus all the more remarkable.[7]

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