Stopping in running and in music performance

Part II. A model of the final ritardando based on runners' deceleration

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Abstract

A model for describing the change of tempo in final ritardandi is presented. The model was based on the previous finding that runners' average deceleration can be characterised by a constant brake power. This implies that velocity is as a square-root function of time or alternatively, a cubic-root function of position. The translation of physical motion to musical tempo is realised by assuming that velocity and musical tempo are equivalent. To account for the variation observed in individual measured ritardandi and in individual decelerations, two parameters were introduced; (1) the parameter q controlling the curvature with q=3 corresponding to the runners' deceleration, and (2) the parameter v_{end} corresponding to the final tempo. A listening experiment gave highest ratings for q=2 and q=3 and lower ratings for higher and lower q values. Out of three tempo functions, the model produced the best fit to individual measured ritardandi and individual decelerations. A commonly used function for modelling tempo variations in phrases (duration is a quadratic function of score position) produced the lowest ratings in the listening experiment and the least good fit to the measured individual ritardandi. The fact that the same model can be used for describing velocity curves in decelerations as well as tempo curves in music provides a striking example of analogies between motion and music.

Introduction

In a previous article we measured the velocity of runners as they stopped from running (Friberg & Sundberg, 1997). Discarding the samples that received the lowest mean rating for "aesthetical quality" by an expert panel, it was found that the average velocity curve was very similar to the average tempo curve of final ritardando (Sundberg & Verillo, 1980). It was also found that the runner kept the decelerating power approximately constant during the entire deceleration. This article presents a model of the final ritardando which is based on this average velocity curve, and complemented by two parameters. The parameters were needed to account for the variation occurring in individual ritardandi observed in music performances. The resulting set of curves is evaluated both in a listening experiment and by fitting the model parameters to measured ritardandi and to runners' decelerations.

Representation of tempo

The choice of dependent and independent variable is obviously crucial in an attempt to elaborate a model for tempo variation. Different strategies for representing tempo data have been used in the past.

The choice of independent variable has also varied. Most researchers have used *score position*. Thus, each note value is expressed in terms of the number of shortest unit, for example 16'th notes, so that the duration of a quarter note is expressed as four units. Another possibility for the independent variable is *time*. This was recommended by Todd (1995) who suggested that ongoing time is more easily perceived than the more abstract score position. An advantage of using time is that tempo curves then tend to assume simpler forms. There is, however, a computational problem. To be applicable to a music example, an arbitrary tempo curve expressed as function of time must be transformed into a function of score position. Unfortunately, such transformations have an analytic solution only for some simple cases.

As dependent variable *tempo,* defined as the inverse of tone duration, seems a natural choice. However, also *beat* or *tone duration* has been used.

(a) Quadratic duration of x

(b) Linear tempo of x

(c) Square root tempo of x
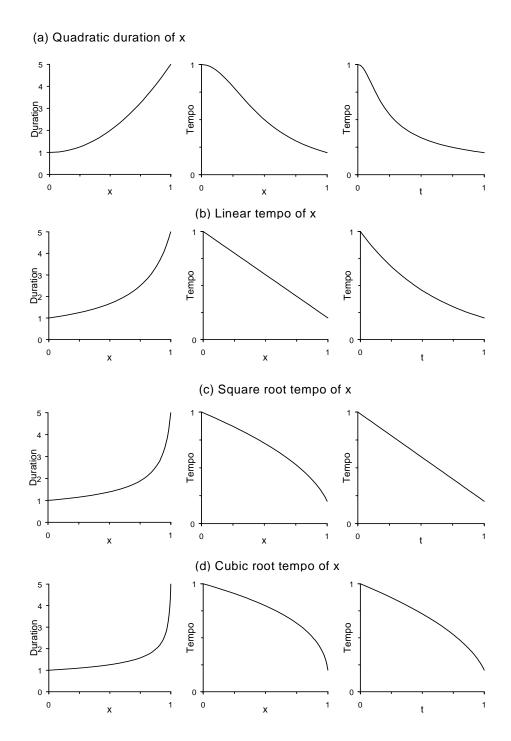
(d) Cubic root tempo of x

Figure 1. A comparison of different tempo representations for four simple final ritardando curves. Duration as a function of position (x) (left) compared with tempo as a function of position (middle) or tempo as a function of time (t) (right).

Figure 1 shows some simple ritardando curves and how they appear in three different representations. The left column of panels in Figure 1 shows duration as a function of score position x, the middle columns of panels shows the same relations transformed into tempo as function of score position, and the right column of panels shows the same relations expressed as tempo as function of time. The choice of variables obviously affect the shape of the curve profoundly. A change of the independent variable from score position to time (mid and right columns) results in an more concave or less convex curvature. For example, when the tempo is a square root function of score position, it becomes a linear function of time. A quadratic relation between duration and score position (top left panel) has been commonly used in the

past. Note that this curve, when transformed into a curve showing tempo versus score position (top middle panel), starts with a convex and ends with a concave curvature.

Previous models

Sundberg & Verrillo (1980) measured the note durations of 24 final ritardandi in phonograph recordings of baroque music. Out of these data they derived a model consisting of two phases, each of which showed a linear decrease of tempo when expressed as function of score position. The length of the second phase corresponded to the last musical motif of the piece.

Several attempts have been made to derive musical tempo variations from physical motion. An early example was provided by Kronman & Sundberg (1987) who used a model of a runner's deceleration. Thereby, assuming that steps corresponded to beats, they hypothesised that step length and deceleration force remain constant throughout the entire deceleration. This implied that the tempo curve was a square root function of score position. This appealing analogy between steps and beats had later to be abandoned, however, when confronted with measurements on real runner's decelerations (Friberg & Sundberg, 1997). (As we have explained elsewhere (op. cit.), the analogy between a runner's deceleration and the final ritardando that was presented by Kronman and Sundberg was based on an erroneous substitution of the independent variable in the Sundberg and Verrillo's investigation, score position instead of normalised time.)

Feldman et al. (1992) investigated curves of performed accelerandi and ritardandi in five examples that were selected by David Epstein from commercially available recordings. The authors developed a simple force model of physical motion which assumed tempo to be equivalent with velocity. They regarded smooth beginnings and endings important characteristics of such tempo changes, and therefore proposed that tempo should be expressed as a quadratic or cubic function of time. These functions correspond to a linear and a quadratic change of force with time, respectively. However, in the subsequent analysis they used beat duration as a function of score position instead of tempo as a function of time. Yet, they fitted their data, expressed in this new form, to linear, quadratic, or cubic functions. They found the two latter alternatives reasonably appropriate to approximate these data and concluded that their

force model neatly accounted for the observed tempo profiles.

Repp (1992a) measured the timing of 28 performances of Robert Schumann's *Träumerei*. The accelerando-ritardando shape of a salient, six-note motif was successfully modelled by expressing tone duration as a quadratic function of score position. He also carried out a perceptual evaluation of this function applied to synthesised piano performances of the same music example (Repp, 1992b).

Todd (1985) presented a model of phrase related tempo changes in music performances. In this model, tone durations were expressed as quadratic functions of score position. Later he proposed a modified version (Todd, 1995) based on an analogy between velocity in physical motion and tempo in music performance, thus implying equivalence between physical position and score position. He simply assumed the deceleration and acceleration forces to be constant which implied linear variations of tempo as a function of time. It can be noted that this case corresponds to a square root function of score position (Figure 1c, middle and right panels), thus similar to the model presented by Kronman & Sundberg (1987).

Building on Todd's idea of a quadratic relation between tempo and score position co-author AF developed a model which related tempo changes to the hierarchical phrase structure (Friberg, 1995a). The model allowed for variations that typically can be observed between different experts' performances of the same piece. Thus, when tested against Repp's measurements of the 28 performances of *Träumerei,* the model successfully accounted for the phrasing variation observed among these performances.

Summarising, the most fruitful analogy between music performance and physical motion is to assume that musical tempo is equivalent to velocity. Furthermore, the musical rubato has been described (1) as a quadratic change of duration versus score position, or (2) as a linear change of tempo versus time whilst the data from running suggest the alternative that (3) tempo is a square root function of time (Friberg & Sundberg, 1997); it is unclear which description is more accurate.

Model

Basic for the construction of our model for final ritardandi was the assumption that the brake power was constant in a runner's deceleration. This implies that the kinetic energy is a linear

function of time and, since kinetic energy is proportional to velocity squared, velocity will be a square root function of time. Let v be velocity (tempo), x be position (score position), then

$$v = \frac{dx}{dt} \sim \sqrt{t} \,. \qquad (1)$$

Primarily for practical purposes, x was chosen as the independent variable. Integrating v, solving for t, and substituting t in (1) we obtain

$$\int dx \sim t^{\frac{3}{2}} \Rightarrow t \sim x^{\frac{2}{3}} \Rightarrow v(x) \sim x^{\frac{1}{3}}$$

Thus, velocity (tempo) is proportional to the cubic root of position (score position). Re-inspection of the individual ritardandi analyzed by Sundberg & Verillo (1980) revealed variation with regard to the overall curvature. To account for this variation, a curvature parameter q was introduced:

$$v(x) \sim x^{\frac{1}{q}}$$

By changing the constant *q* a number of different curvatures can be obtained, including

the runners' mean deceleration (q=3) as well as the previously mentioned square-root function (q=2).

Unlike velocity in locomotion, the tempo never reaches zero in music, if tempo is defined as the inverse of tone duration. This implies the need for a second parameter, the final tempo v_{end}. The resulting model of the tempo (v) as a function of score position (x) was defined as

$$v(x) = \left[1 + (v_{end}^q - 1)x\right]^{\frac{1}{q}} \qquad (2)$$

Here, tempo and v_{end} are normalized with respect to pre-ritardando tempo (v_{pre}) and position is normalized with respect to total ritardando length measured in score units. Score position x=1 occurs at the onset of the last note. Figure 2 shows the resulting tempo curves for four different q values. The values of q=1, q=2, and q=3 correspond to (a) a linear function of x, (b) a square root function of x (i.e., a linear function of t), as proposed earlier by Kronman & Sundberg and Todd, and (c) the approximation of runners' deceleration, respectively.
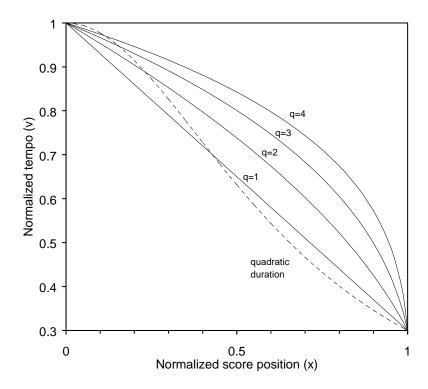

Figure 2. The ritardando model with four different q values (q=1,2,3,4) and the quadratic duration function. The final tempo was fixed at v_{end}. These were the five curves used in the listening experiment.

Table 1. Music examples, preritardando tempi (v_{pre}) and the final tempo values (v_{end}) used in the listening experiment. Note that v_{end} is normalized with respect to v_{pre}.

Music example	Abbr.	v_{pre} (shortest notes/s)	v_{end}
J. S. Bach: Eng. Suite 2 Prel., last 5 meas.	E2P	6.48	0.3
J. S. Bach: Wohltemp. clav. I Prel. 1, last 5 meas.	WlP	4.47	0.4
minor second sequence (Db4-C4-Db4-C4-...Db4), 21 notes totally	m2	5	0.4

The translation from the continuous curve to discrete tones is realized by integrating the inverse of the tempo function (Eq. 2) (see Todd, submitted).

$$\int \frac{1}{v} = \int \frac{dt}{dx} = \int \left[1 + (v_{end}^q - 1)x\right]^{-\frac{1}{q}}$$

Let $k = v_{end}^q - 1$. Then, we obtain time as function of score position:

$$t(x) = \frac{q(1+kx)^{\frac{q-1}{q}} - q}{(q-1)k}, \; q > 1 \qquad (3)$$

The integration constant was determined by setting $t(0) = 0$. The duration (DR) of a tone is given by the time difference for the x values corresponding to the onset x_{on} and offset x_{off} of the tone.

$$DR = t(x_{off}) - t(x_{on}) =$$

$$\qquad (4)$$

$$\frac{q(1+kx_{off})^{\frac{q-1}{q}} - q(1+kx_{on})^{\frac{q-1}{q}}}{(q-1)k}$$

The advantage of using $t(x)$ to determine the tone durations is that it is independent of note values. For example, four sixteenth notes will add up exactly to the duration of one quarter note.

The model (Eq. 2) has the advantage that it is easy to transform, if time instead of position is preferred as the independent variable. Tempo as a function of time can be obtained by solving (3) for x and substituting the result in (2):

$$v(t) = \left[1 + (v_{end}^{q-1} - 1)t\right]^{\frac{1}{q-1}}$$

Note that this equation is essentially the same as equation (2) with the value of q decreased by one.

Perceptual evaluation

A listening experiment was performed to assess the preferred curvature value q in different music examples. The previously mentioned phrasing curve with duration as a quadratic function of score position was included as an additional alternative.

Method

Stimuli and procedure

Three different music examples were used, two excerpts from pieces by J.S. Bach and a sequence of two alternating notes, a minor second apart, see Table 1. The purpose of the Bach examples was to have two different musically realistic examples. The minor second example was chosen to attain a minimum of musical content without destroying the perception of the ritardando curve; in our informal pretests, we found that a simple tone repetition was not enough to differentiate the curvatures.

Four different q values (q=1,2,3,4) were used for the ritardando model (Eq. 2). For the phrasing curve with duration as a quadratic function of score position the expression $v(x) = 1/(0.7x^2/v_{end}+1)$ was used.

The preritardando tempo (v_{pre}), ritardando length and ritardando depth (v_{end}) were fixed for each music example throughout the test, see Table 1. These values were determined in collaboration with two professional musicians, Lars Frydén and Monika Thomasson. In all examples, the ritardando started at the first note onset occurring at least 1300 ms, as measured in

preritardando tempo, before the onset of the last note.

In the pretests it was found that the duration of the penultimate note seemed important to the perceived depth of the ritardando. The method described above for translation of the continuous tempo function to discrete note durations did not permit the setting of an exact value for the lengthening of the penultimate note. Therefore, a second normalization was performed so that the duration of the penultimate note was set to $1/(v_{end}*v_{pre})$.

Special attention had to be paid to the last note. Its note value is sometimes the same as that of the preceding note but sometimes considerably longer. In the latter case, no further prolongation of the last note is called for. On the contrary, such long final notes can even be shortened in a real performance. In the test, the duration of the final note was simply set to 1.25 times the duration of the penultimate note. However, if the last note was already longer than this, it was left unchanged.

To exclude the influence of dynamics, the music examples were played on a sampler synthesizer (SampleCell) using recorded tones from a harpsichord. The ritardando model was implemented in the Director Musices program (Friberg, 1995b) on a Macintosh computer. The examples were recorded on a DAT tape with some artificial reverberation (Yamaha REV7). All subjects listened to the tape over earphones adjusted to a comfortable listening level.

The first six examples on the test tape were used as practice trials. They were followed by a total of 51 test trials (5 curves X 3 music examples X 3 repetitions) in random order. The test took 17 min.

The subjects were asked to mark on a 10 cm long visual analogue scale on an answering sheet how musical they found the performance of the ritardando. The endpoints of the line were labelled "extremely good" and "extremely bad". The listeners were also instructed to try to ignore the performance of the music preceding the ritardando as well as the length of the final note. The rating values used in the subsequent analysis was defined as the length in mm from the "extremely bad" endpoint, i.e. the better ritardando, the higher the rating value.

Subjects

The subjects were 19 students all taking the course in music acoustics at KTH. After the test, they answered a questionnaire about their musical background. The questionnaire asked if they had experience of performing music, if so, what instrument, how many years of performing, how many hours per week and in which music styles. In addition, they could make any general comment about the test.

According to the answers from the questionnaire, most of the subjects could be considered as amateur musicians. The final ritardando is mostly heard in classical music and therefore mainly classical musicians would have practical experience of performing final ritardandi. Nine subjects reported that they had some experience of classical music. One of these had professional experience as a classical musician. Most subjects found the test quite difficult with the exception of the professional player. Thus, the task was difficult for listeners lacking previous experience with performance of ritardandi on a professional level.

Results

Using the subjects' ratings as the dependent variable the results were submitted to a repeated-measures analysis of variance performed by the SuperANOVA 1.11 program for Macintosh. The within-factors were 5 curves X 3 music examples X 3 repetitions. Despite the difficulty of the test, the main effect of curve was highly significant (p<0.0001), indicating that the listeners could clearly differentiate the curves. The main effects of music example and repetition were also significant (p<0.0001 and p<0.015). No interaction terms were significant. The main effect of music example and repetition disappeared, when the m2 example was excluded from the analysis. The fact that there was a strong effect of repetition only in the case of the m2 example may indicate that this example had a strange musical content that, however, the listeners gradually got used to.

The mean ratings for the different curvatures are shown in Figure 3. They reveal that the two curvatures that received the highest ratings originated from the model with q=2 and q=3. The first value corresponds to the model supported by Kronman & Sundberg (1987) and by Todd (1995), while the second value corresponds to the curvature derived from runners' decelerations. Both for higher and lower q values and the quadratic duration, the mean ratings were lower as indicated by a highly significant contrast analysis resulting from a comparison of the cases q=2 and q=3 with the three remaining alternatives (p<0.0001).
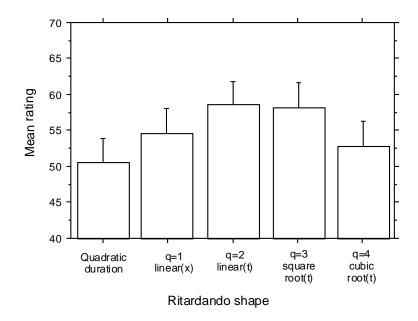
Figure 3. Overall mean ratings of the different curvatures. The error bars indicate 95% confidence intervals.
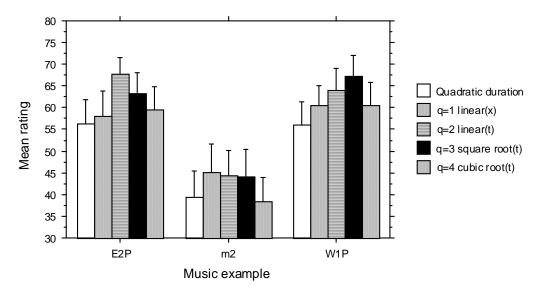

Figure 4. Mean ratings for the interaction of music example and ritardando curve. The bars indicate the positive parts of a 95% confidence interval.

Figure 4, showing the means for each music example and curve, offers a more detailed representation of the results. First, it can be noted that for all curves the minor second example was rated much lower than the other examples. Obviously, despite the instruction to concentrate on the ritardando, the subjects were unable to disregard the musical context. Second, the maximum rating was obtained for the q=2 curve in the E2P example and for the q=3 shape in the W1P example. This suggests that the optimum curve is dependent on the music example. A

contrast analysis comparing q=2 for E2P and q=3 for W1P with q=3 for E2P and q=2 for W1P showed a barely significant difference (p<0.04). Further testing with more skilled listeners is needed to further test this difference.

Matching the model to measurements

The model with q=3, i.e. the runners' deceleration, was previously found to fit well
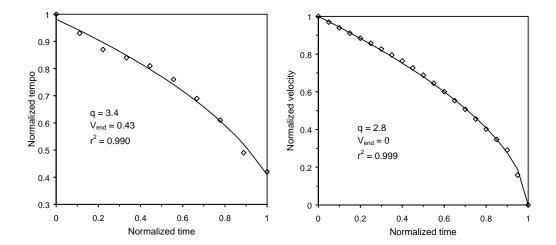
Figure 5. Left panel: The model (line) fitted to the mean ritardando curve (diamonds) from Sundberg and Verillo. Right panel: The model (line) fitted to the mean velocity curve (diamonds) of the runners' deceleration. Note that time rather than position is used as the independent variable.

with the average ritardando shape (Sundberg & Verillo, 1980; Friberg & Sundberg, 1997). In this section, the ability of the model to fit performed mean and individual ritardandi as well as runner's mean and individual decelerations will be examined.

The q and v_{end} parameters in the model, defined in Eq 2, were varied by means of the solver function in Microsoft Excel 7.0, such that the sum of the squared distances between the model and the measured data was minimized. In addition, a third parameter, v_{offset}, was varied in the optimizing process. This parameter simply added a constant to the model.

Average curves

As an initial check of previous results, the model was fitted to the curves for the mean ritardando and the mean deceleration (Friberg & Sundberg, 1997), see Figure 5. The fit is good in both cases. In the case of the deceleration curve, however, this was expected, as the model was originally derived from this curve. Figure 5 also specifies the optimal values of the parameters q and v_{end}. For both cases the q values were close to 3. The small difference between q=3.4 and q=2.8 is probably of minor relevance; while our previous results showed that a q difference of 1.0 was clearly perceptible, an informal listening test suggested that a q difference of 0.6 was not noticeable.

Individual ritardandi measured by Sundberg and Verillo

The model was fitted also to individual ritardandi taken from the original raw data by Sundberg and Verillo. To improve the reliability of the fitted curvature data only ritardandi with a smooth shape, containing a minimum of individual note departures, were selected. Thus, only the performances with at least 6 consecutive final notes with monotonically increasing durations were used. In this way, 12 final ritardandi were selected out of totally 22, all by J.S. Bach (Table 2). Apart from the present model, two other alternatives were tried. As note duration is frequently assumed to be a quadratic function of score position, one alternative was a quadratic polynomial in which the three parameters were fitted to measured durations (henceforth quadratic duration). In the other alternative, a similar quadratic function was used to approximate the tempo rather than note duration (henceforth quadratic tempo). The fitted model parameter values and the resulting determination coefficients (r^2) for the 12 examples are listed in Table 2.

Figure 6 shows the model and the two alternatives fitted to the first music example (WIP). Notice that the quadratic duration has the undesired property of an initial tempo increase.

Table 2. Music examples used in the test where the model was fitted to data for individual ritardandi measured by Sundberg & Verillo (1980), keeping their abbreviations (parentheses). The values of the parameters q, v_{end} and v_{offset} that generated the best fit of the model are also listed. The three rightmost columns show the determination coefficients r^2 for the model and for the two alternatives tried.

Music examples		Fitted model parameters			Squared correlation coefficient (r^2)		
	Notes	q	v_{end}	v_{offset}	Model	Quadratic duration	Quadratic tempo
W. clav I Prel. 1 (WIP)	10	2.5	0.32	0.005	0.982	0.927	0.975
W. clav II Prel. 1 (WP1)	8	2.1	0.37	0.0001	0.980	0.989	0.983
W. clav II Prel. 2 (WP2)	7	2.4	0.51	0.03	0.967	0.989	0.977
W. clav II Fug. 3 (WF3)	6	1.2	0.48	-0.03	0.969	0.951	0.968
W. clav II Fug. 5 a (WF5)	7	4.2	0.50	0.01	0.995	0.970	0.995
W. clav II Fug. 5 b (WF5)	8	2.6	0.37	-0.02	0.975	0.919	0.966
Eng. Suite 1 Allem. (E1A)	6	2.0	0.45	-0.03	0.982	0.943	0.978
Eng. Suite 2 Allem. (E2A)	11	3.7	0.37	0.02	0.975	0.927	0.980
Fr. Suite 4 Allem. (F4A)	6	4.2	0.50	0.02	0.991	0.959	0.986
Fr. Suite 6 Allem. (F6A)	7	2.4	0.45	0.02	0.981	0.965	0.979
Fr. Suite 6 Cour. (F6C)	7	5.0	0.46	-0.008	0.994	0.902	0.959
Italian Conc. Mvt. 3 (IC3)	7	1.2	0.34	-0.01	0.970	0.961	0.970
Mean		2.8	0.43	0.001	0.980	0.950	0.976
SD		1.2	0.07	0.02	0.01	0.03	0.01
Max		5.0	0.51	0.03	0.995	0.989	0.995
Min		1.2	0.32	-0.03	0.967	0.902	0.959

As shown in Table 2 the mean q value (q=2.8) is very close to the initial value of q=3 that was derived from the runners' decelerations. Note that example WIP was also used in the listening experiment above. It received a q value of 2.5 which is just in the middle of the listeners' preference, as seen in Figure 3. However, q varied substantially between examples. Indeed, two examples yielded q=1.2, i.e., the tempo decreased almost linearly with score position and in three examples q exceeded the value of four. Such low and high values of q received low ratings in the listening test above, and were yet obviously acceptable in these performances. This supports the assumption above that the optimal ritardando curve depends on some characteristics of the music example, e.g., musical structure and tempo. Another factor of potential relevance is the performer's intention or preference. The final tempo v_{end} varied comparatively less, between 32% and 51% of the initial tempo.

The model produced the highest mean correlation with the measurements (mean r^2 = 0.98). A t-test comparing the mean r^2 for the model and the mean r^2 for the quadratic duration alternative yielded a significant difference (p<0.003): this means that, on average, the model approximated the measured ritardandi better than the quadratic duration.

Although, the correlation was slightly better for the model than for the quadratic tempo polynomial, this difference was not significant. With regard to the shortest ritardandi, consisting of no more than six notes, it could be argued that many slightly curved functions would offer a good fit. Longer ritardandi, on the other hand, often exhibited a more pronounced tempo decrease in the end than in the beginning. This case can be accounted for by the model by means of a relatively high value of q (Figure 7). The quadratic tempo function fails to produce the characteristic increased curvature in the end of the ritardando.
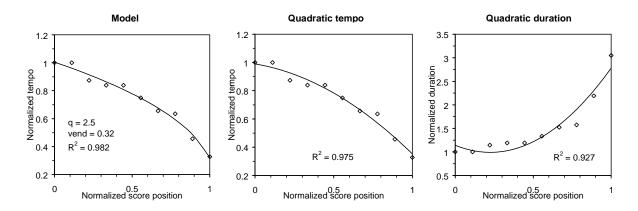
Figure 6. Model(left), quadratic tempo (middle) and quadratic duration (right) functions fitted to a measured performance of the music example WIP.
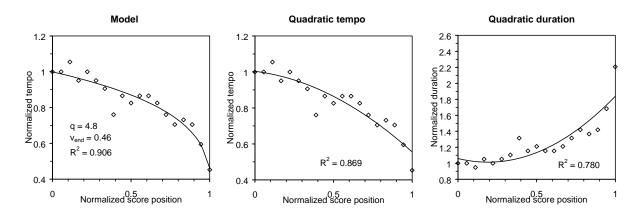

Figure 7. Model (left), quadratic tempo (middle) and quadratic duration (right) functions fitted to a measured performance of a comparatively long ritardando (music example WP19 in Sundberg and Verillo, 1980, not included in test referred to in Table 2). The increased curvature in the end of the ritardando can not be modelled with a quadratic tempo function as seen in the figure.
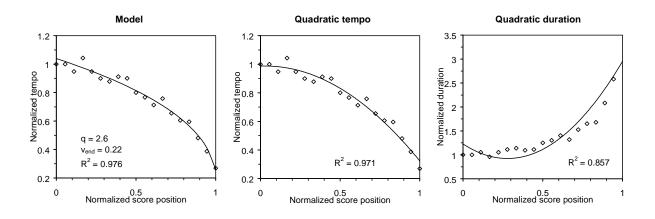

Figure 8. Model(left), quadratic tempo (middle) and quadratic duration (right) functions fitted to a ritardando measured by Feldman et al. (1992).

Table 3. Results of attempts to fit the model and a quadratic velocity function to each of the 12 decelerations used by the authors (1997) for calculating a mean deceleration curve. The two rightmost columns show the determination coefficients r^2 for the model and for the quadratic velocity function.

	q	v_{end}	v_{offset}	Model r^2	Quadratic velocity r^2
Mean	2.8	0.27	0.004	0.990	0.9778
SD	0.5	0.08	0.02	0.01	0.02
Max	3.4	0.39	0.03	0.999	0.9995
Min	1.8	0.13	-0.02	0.978	0.939

Individual ritardandi measured by Feldman, Epstein and Richards

Feldman et al. (1992) measured two ritardandi in performances of orchestral music. These ritardandi occurred within the pieces, i.e. not in a final position. Also, they were clearly longer than the final ritardandi considered above, and were measured on the beat level instead on the note level. The same three functions as above were fitted to the example (Sample 1: A Dvorak, Slavonic Dance, op 48:8, measures 243-272), which had a smoothest shape (Figure 8). Despite the difference in style, length and context, the result is similar to that observed in Figure 7. The model produced the best fit, slightly better than that of the quadratic tempo function, especially in the end of the ritardando. The fit obtained from the quadratic duration gave the least good fit.

Individual decelerations of runners

Friberg & Sundberg (1997) used 12 selected decelerations for computing the average velocity curve for runners' deceleration. The model and a quadratic velocity function was fitted also to each of these 12 decelerations. Table 3 shows the results in terms of averages across examples.

Table 3 shows that, again, the model produced a better fit than the quadratic velocity alternative. According to a two-tailed t-test, the means of the determination coefficients r^2 differed in this case significantly (p<0.02) between the model and the quadratic velocity function.

Discussion

In discussing the model proposed here for description of final ritardandi and runners' decelerations, certain limitations should be born in mind. The start of the final ritardando is difficult to identify in measurements. Perceptually a smooth onset of the ritardando is important, as mentioned. For example, informal listening tests revealed that the onset of the ritardando was far too abrupt when the alternative linear tempo function of score position was used. Obviously, a smooth onset complicates the identification of the starting point. On the other hand, a rigorous method was applied for identifying the ritardando onsets for the individual ritardandi referred to in Table 2: all tones following the ritardando onset show a progressively decreasing tempo. This definition favours late ritardando onsets. Such cases tend to reduce the value of q.

There are also other limitations. The perception of a final ritardando would depend not only on tempo changes but also on dynamic changes. This parameter was not analyzed in the present investigation. Another important factor is its total length. What is considered an appropriate ritardando may also depend on the instrument played. In view of the analogies suggested by our finding between music and locomotion, it would be tempting to explore possible musical equivalents of e.g. inertia and its effect on the final ritardando.

It appeared that it was more difficult for the listeners to differentiate the tempo curves in a musically unnatural example. In selecting the music examples for the listening test, the authors frequently noted that the perceived character of the different curves were well exposed by real music examples and almost impossible to distinguish in unrealistic, simple examples, such as a sequence of tone repetitions. On the other hand, the tempo curve seems to originate from the paramount experience of locomotion. Maybe the musical unnaturalness of such examples distracts the listener's attention from the tempo curve. In any event, realistic musical examples seem crucial for a correct evaluation of any music performance model.

The quadratic duration curve gave the poorest results, both in the listening experiment and in the curve fittings. This was surprising since it had been successfully applied to describe tempo curves associated with phrasing (Todd, 1985; Repp, 1992; Friberg, 1995a; Penel

& Drake, in press). Transformed to tempo as function of score position the quadratic duration curve assumes a shape characterized by a gradual decrease of the curve steepness, as was illustrated in Figure 1a. If translated to velocity in locomotion such a curve shape would imply that the runner refrains from spending energy on reduction of speed toward the end of the deceleration process, thus suggesting a continuation of the movement. This message appears quite appropriate at phrase endings to a music listener; the music continues beyond the phrase boundary. Indeed, according to informal listening tests the impression of an approaching final stop disappeared when the quadratic duration curve was used for final ritardandi; it was no more obvious that the last tone really was the last.

Surprisingly, we found two cases of q=1.2 in the fittings of the individual ritardandi, i.e., the tempo decreased almost linearly with score position. Such a curve (q=1) was rated low in the listening experiment. Also, when applied in informal tests to a variety of music examples, this curve sounded musically unacceptable. As with the quadratic duration, the problem was that the piece did not appear to approach a final stop during the last part of the ritardando.

A factor of possible relevance to our results is musical style. The measurements and fittings in which the quadratic duration was used mainly concerned romantic classical music while the present final ritardando model was mostly tested on Baroque music (except for Sample 1 from Feldman et al. (1992) which was composed by Dvorak). It is possible that the poor success of the quadratic duration alternative reflected that these two music styles are associated with different types of motion. It is also possible that the final ritardando model would work as a description of the tempo modulation in phrases. We plan to check this possibility in future research.

Why does our model work? The two cases q=2 and q=3, which received the highest ratings in the listening experiment, have a very simple form in terms of locomotion; the former implies that the breaking force is constant while the latter implies that the breaking power is constant. Such simple relations would facilitate prediction of the final stop.

Conclusion

The present model of the final ritardando was based on the previous finding that the braking power is approximately constant during runners' decelerations (Friberg & Sundberg, 1997). By introducing two parameters, q for curvature and v_{end} for the end value, the model could well describe the average final ritardando tempo curve, the average runners' decelerations velocity curve, individual final ritardandi, and individual decelerations. These findings substantiate the common assumption that locomotion and music are related.

The listening experiment indicated a preference for q values between $2 \le q \le 3$, where q=3 corresponds to the average velocity of the runners' decelerations and q=2 to a previously proposed model for final ritardandi (Kronman & Sundberg, 1987) as well as to a model for phrase related tempo curves presented by Todd (1995). However, the curvature range was larger in the individual ritardandi than in the listening experiment, thus suggesting influence of musical factors. Describing duration as a quadratic function of score position, previously found applicable to tempo modulations occurring in phrases, was tried with poor results both in a listening test and in attempts to match individual measured ritardandi.

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